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☆ Classes (sets) Collection of objects.

*** Membership** $a \in A$ "a belongs to A"

(member) (class)

$a \notin A$ "a does not belong to A"

$A = \{1, 2, 3\}$

*** Inclusion** $A \subseteq B$ Members of Class A all belong as members of Class B

(Subclsss) (Superclass)

- If $A \subseteq B$ and $B \subseteq A$, then $A = B$
- Empty class : 0 ← subclass of every class
- Universal Class : 1 ← any class a subclass of

A' "Complement of A"

$0' = 1$ $1' = 0$

*** Union**

$A \cup B$

("or")

*** Intersection**

$A \cap B$

("and")

Example (pg.5)

"If A is the class of green-eyed cats, and B is the class of long-haired cats, then $A \cap B$ is the class of long-haired green-eyed cats."

$A \cup 0 = A$ $A \cup 1 = 1$ $A \cup A' = 1$

$A \cap 1 = A$ $A \cap 0 = 0$ $A \cap A' = 0$

Some Important Relations

$A \cap B \subseteq A$ $A \cap B \subseteq B$

$A \subseteq A \cup B$ $B \subseteq A \cup B$

Example Proof

- These 3 relations are equivalent.

① $A \subseteq B$, ② $A \cup B = B$, ③ $A \cap B = A$.

Assume ① is true.

- It follows any member of $A \cup B$ is thereby a member of B.

  $\Rightarrow A \cup B \subseteq B$
- But, B must be a member of $A \cup B$.

  $\Rightarrow B \subseteq A \cup B$

Hence, ② follows.

Assume ② is true.

- Since $A \subseteq A \cup B$ and $A \cup B = B$, it follows any member of A is a member of B.

Hence, ① follows. $\Rightarrow A \subseteq B$

Assume ③ is true.

- Since $A \cap B \subseteq B$ and $A \cap B = A$, any member of A must be a member of B.

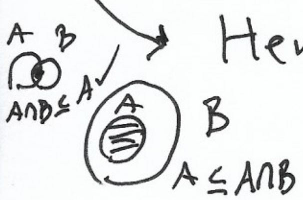
Hence, ① follows. $\Rightarrow A \subseteq B$

Assume ① is true.

- Since $A \cap B \subseteq A$ and $A \subseteq B$, it follows any member of A is a member of $A \cap B$.

Hence, ③ follows $\Rightarrow A \cap B \subseteq A$ and $A \subseteq A \cap B$

So, $A \cap B = A$



De Morgan $(A\cup B)' = A'\cap B'$ $(A\cap B)' = A'\cup B'$

Proof

If $x \in (A\cup B)'$, then $x \notin A\cup B$ and $x \notin A$ and $x \notin B$.
This implies $x \in A'$ and $x \in B'$.
Hence, $x \in A'\cap B'$.
And thus, $(A\cup B)' \subseteq A'\cap B'$.
However, if $x \in A'\cap B'$, then $x \in A'$ and $x \in B'$.
That is, $x \notin A$ and $x \notin B$ and, thus, $x \notin A\cup B$.
But because $x \notin A\cup B$, $x \in (A\cup B)'$.
And thus, $A'\cap B' \subseteq (A\cup B)'$.
Conclusion: $(A\cup B)' = A'\cap B'$ ∎

Proof via Complementation

Given $(A\cup B)' = A'\cap B'$,
Substitute A' for A and B' for B.
$\Rightarrow (A'\cup B')' = A''\cap B'' \Rightarrow (A'\cup B')' = A\cap B$
Then complement both sides.
$\Rightarrow (A'\cup B')'' = (A\cap B)' \Rightarrow A'\cup B' = (A\cap B)'$ ∎

Associative Law

$A\cup(B\cup C) = (A\cup B)\cup C$ $A\cap(B\cap C) = (A\cap B)\cap C$

- Extends to any number of classes.
- Suffices to observe is true

Distributive Laws

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $\quad$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: If $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$.
So, if the former, then $x \in A \cup B$ and $x \in A \cup C$.
That would imply $x \in (A \cup B) \cap (A \cup C)$.
And, if the latter, then $x \in B$ and $x \in C$.
That would imply $x \in A \cup B$ and $x \in A \cup C$,
which thereby means $x \in (A \cup B) \cap (A \cup C)$.
$\Rightarrow$ Thus, $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$.
Conversely, if $x \in (A \cup B) \cap (A \cup C)$, then $x \in A \cup B$ and $x \in A \cup C$.
If $x \notin A$, then necessarily $x \in B$ and $x \in C$ such that
$x \in B \cap C$ and so $x \in A \cup (B \cap C)$.
And if $x \in A$, then it's still true $x \in A \cup (B \cap C)$.
$\Rightarrow$ Thus, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Conclusion: $\therefore A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ ■

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* Difference

$A - B$ — The class of all elements of A which are not elements of B.

$A - B = A \cap B'$ (follows from def^n)

Observation... $1 - A = A'$ (for $1 - A = 1 \cap A' = A'$)

$$A - B = 0 \leftrightarrow A \subseteq B$$

$(A - B) \cup B = A \cup B$

for $(A - B) \cup B = (A \cap B') \cup B = (A \cup B) \cap (B' \cup B)$

$= (A \cup B) \cap 1 = A \cup B$ ■

Distributive Laws

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof If $x \in A \cup (B \cap C)$, then $x \in A$ or $x \in B \cap C$.
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$(A - B) \cup B = A \cup B$
for $(A-B) \cup B = (A \cap B') \cup B = (A \cup B) \cap (B' \cup B)$
$= (A \cup B) \cap 1 = A \cup B$ ∎

Again, intersection distributes over symmetric differences.

$C \cap (A+B) = (C \cap A) + (C \cap B)$

Proof) $C \cap (A+B) = C \cap \{(A-B) \cup (B-A)\}$

$= \{C \cap (A-B)\} \cup \{C \cap (B-A)\}$

$= \{(C \cap A) - (C \cap B)\} \cup \{(C \cap B) - (C \cap A)\}$

$= (C \cap A) + (C \cap B)$ ∎

But the union doesn't distribute.

For Example) $A \cup (A+B) = A \cup B$

But, $(A \cup A) + (A \cup B) = A + (A \cup B) = B \cap A'$ X

* Cross

$A \times B$ the complement of $A+B$ such that

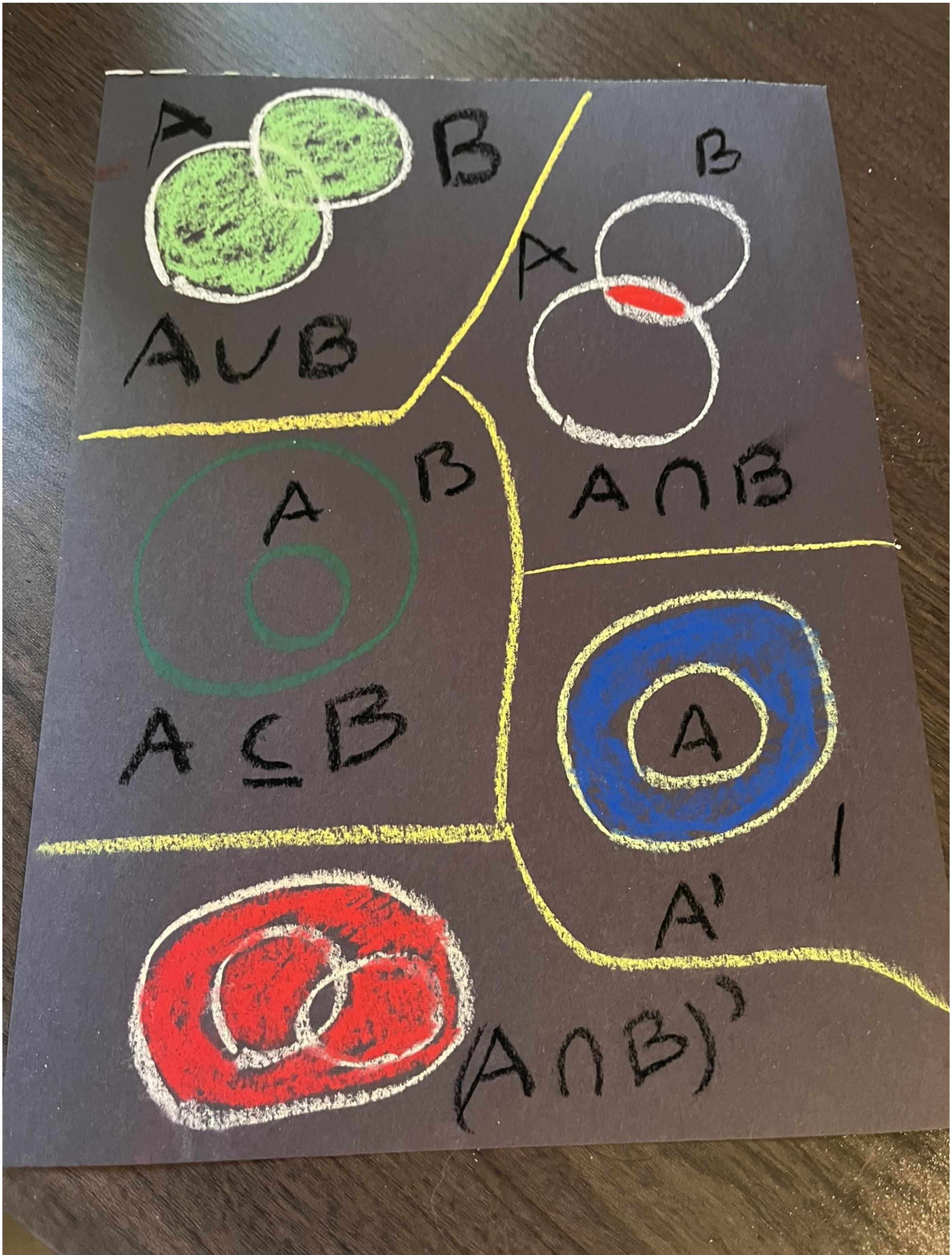
$A \times B = (A \cap B) \cup (A' \cap B')$

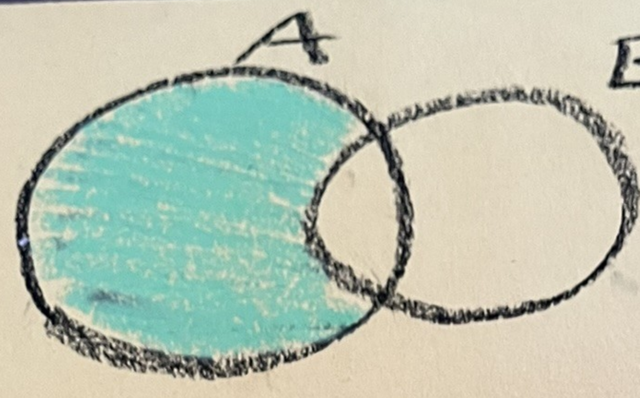
From that $A' \times B' = A \times B$

Remember, symmetric difference is commutative where $B + A = (B-A) \cup (A-B) = (A-B) \cup (B-A) = A+B$

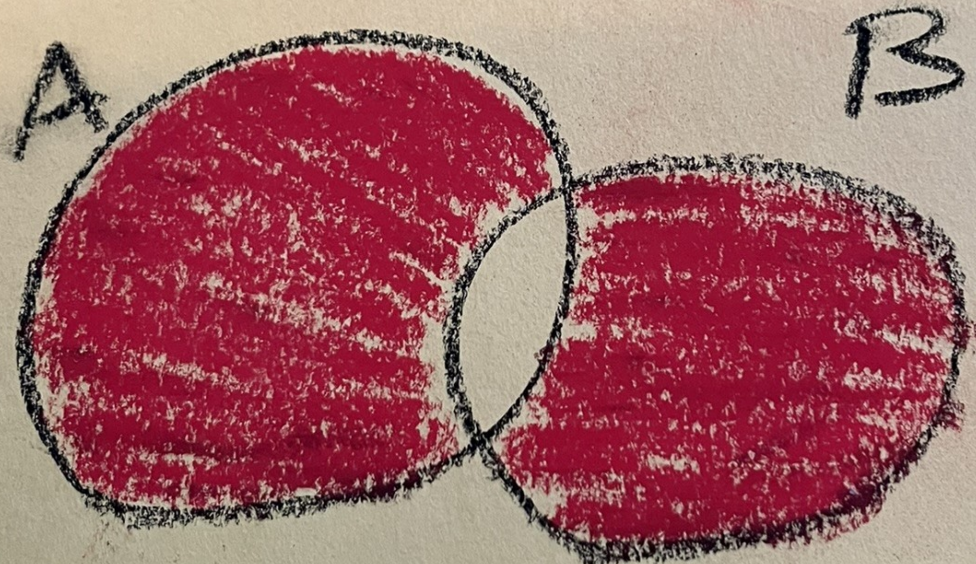
By taking complements $A \times B = B \times A$,

$(A \times B) \times C = A \times (B \times C)$.


A
B
A∪B
B
A
A∩B
B
A
A⊆B
A
A'
(A∩B)'



A-B



A+B